Ascham School

Student Number ______________________

Teacher **AF**

2023 HSC Trial Examination

Mathematics Extension 2

Reading time - 10 minutes

Working time - 3 hours

General Instructions

- Write using black pen.
- Diagrams drawn using dark pencil.
- A NESA-approved calculator may be used.
- For questions in **Section** II, All relevant working should be shown for each question in the answer booklets provided.

Additional Materials Needed

- Reference Sheet

Structure & Suggested Time Spent

- **Section I** (Multiple Choice) **10 Marks**
  Attempt Questions 1-10
  Allow about 15 minutes for this section.
- **Section II** (Extended Response) 9**0 Marks**
  Attempt Questions 11-16
  Answer in the booklets provided.
  Allow about 2 hours 45 minutes for this section.

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

1 Evaluate $i^{2023}+i^{2022}+i^{2021}$.

A. $-i$

B. -1

C. 0

D. 1

2 Consider the following statement: "If I play chess, then I am a girl."
Which statement is the contrapositive?

A. "If I do not play chess, then I am not a girl."

B. "If I am not a girl, then I am not good at chess."

C. "If I am not a girl, then I do not play chess."

D. "If I am a girl, then I play chess."

3 Which of the following is a vector equation of the line joining the points $A(1,3)$ and $B(-2,4)$?

A. $\underset{\sim}{r}=\begin{pmatrix}1\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\4\end{pmatrix}$

B. $\underset{\sim}{r}=\begin{pmatrix}-2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\3\end{pmatrix}$

C. $\underset{\sim}{r}=\begin{pmatrix}1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\1\end{pmatrix}$

D. $\underset{\sim}{r}=\begin{pmatrix}1\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\end{pmatrix}$

4 Two vectors are such that $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$.

What is the correct evaluation of $|\underset{\sim}{u}| \times \underset{\sim}{v} \bullet \underset{\sim}{v}$?

A. $\begin{pmatrix} \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}$

B. $\begin{pmatrix} 8\sqrt{14} \\ 4\sqrt{14} \\ 8\sqrt{14} \end{pmatrix}$

C. $36\sqrt{14}$

D. 84

5 A variable force acts on a particle causing it to move in a straight line.
At time t seconds its velocity v metres per second and position x metres from the origin are such that $v = e^x \cos x$.
The acceleration of the particle can be expressed as

A. $-e^x \sin x$

B. $e^{2x} \cos x(\cos x - \sin x)$

C. $e^x(\sin x + \cos x)$

D. $2e^{2x}(\cos^2 x)$

6 Consider the statement: $n^2 - n \geq 0$ for all positive integers n. Which of the following is true?

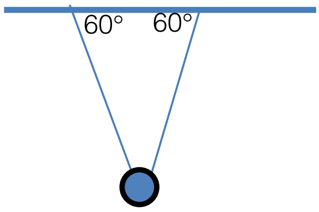
A. The statement cannot be disproven with a counterexample.

B. The statement can be disproven with the counterexample when $n = 1$.

C. The statement can be disproven with the counterexample when $n = \frac{1}{2}$.

D. The statement can be disproven with the counterexample when $n = 0$.

7 A ball of mass 12 *kg* is suspended from a horizontal ceiling by two identical light strings.
Each string makes an angle of $60°$ with the ceiling as shown.
What is the magnitude in Newtons of the tension in each string?



A. $6g$

B. $12\,g$

C. $24\,g$

D. $4\sqrt{3}\,g$

8 The integral $\int_{-1}^{1}\left(\frac{1}{x\sqrt{4+x^2}}\right)dx$ can be most efficiently solved with which substitution?

A. $x=\tan\alpha$

B. $x=\sec\alpha$

C. $x=2\sec\alpha$

D. $x=2\tan\alpha$

9 The algebraic fraction $\frac{7x-5}{(x-4)^2(x^2+9)}$ is decomposed into partial fractions.

What is the correct decomposition?

A. $\frac{A}{(x-4)^2}+\frac{B}{x-4}+\frac{Cx+D}{x^2+9}$

B. $\frac{A}{(x-4)^2}+\frac{Bx+C}{x^2+9}$

C. $\frac{A}{x-4}+\frac{B}{x+3}+\frac{C}{x-3}$

D. $\frac{A}{(x-4)^2}+\frac{B}{x-4}+\frac{C}{x^2+9}$

10 On an Argand diagram, a set of points lies on a circle of radius 2, centred at the origin. Which of the following defines this circle?

A. $\{z \in \mathbb{C} : z\overline{z} = 2\}$

B. $\{z \in \mathbb{C} : z^2 = 4\}$

C. $\{z \in \mathbb{C} : \mathrm{Re}(z^2) + \mathrm{Im}(z^2) = 4\}$

D. $\{z \in \mathbb{C} : (z + \overline{z})^2 - (z - \overline{z})^2 = 16\}$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a separate writing booklet.

(a) Find the acute angle (to the nearest degree) between the vectors **2**

$$\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \text{ and } \underset{\sim}{v} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

(b) Solve $z^2 - z + 7 = 0$ over the complex plane. **3**

(c) Relative to a fixed origin O, the point A has position vector $2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$, the point B has position vector $4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and the point C has position vector $a\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$, where a is a constant and $a > 0$. D is the point such that $\overrightarrow{AB} = \overrightarrow{BD}$.

(i) Find the position vector of D. **2**

(ii) If $\left|\overrightarrow{AC}\right| = 4$, find the value of a. **3**

(d) A particle is moving on a straight line. Its velocity v is given by $v^2 = 4\left(2x - x^2\right)$ where x is its displacement from a point O on the line.

(i) Show that its acceleration is given by $\ddot{x} = -4\left(x - 1\right)$. **2**

(ii) Explain why this particle moves in simple harmonic motion. **1**

(iii) Find the maximum speed of the particle. **2**

Question 12 (15 marks) Use a separate writing booklet.

(a) (i) Express $\dfrac{x^2+1}{(x-1)(x+2)(x^2+x+1)}$ as the sum of partial fractions. **4**

(ii) Hence, or otherwise, find $\displaystyle\int \frac{x^2+1}{(x-1)(x+2)(x^2+x+1)}dx$. **3**

(b) A triangle has side lengths $x, 5, 6$. What are the possible values of x? **1**

(c) Simplify $\dfrac{3+i}{7-2i}$ and hence state its real and imaginary parts. **3**

(d) Factorise $2z^6+z^4-2z^2-1=0$ fully into its complex factors. **4**

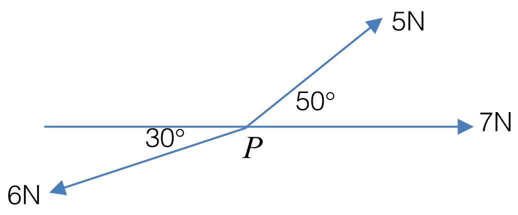
Question 13 (15 marks) Use a separate writing booklet.

(a) Find $\int sec x\, dx$ by using the substitution $t=\tan\left(\dfrac{x}{2}\right)$. **3**

(b) (i) Express $z=\sqrt{3}+i$ in the form $re^{i\theta}$. **2**

(ii) Hence, or otherwise, simplify z^{18} **2**

(c) Three coplanar forces act at a point P. The magnitudes are 5N, 6N and 7N. **4**
The directions in which the forces act are as shown in the diagram.
Find the magnitude and the direction of the resultant of the three forces.
Give the magnitude correct to four significant figures and the direction correct to the nearest degree.



(d) A sphere has a centre at (3,–3,4) and its radius is 6 units.

A line has equation $\underset{\sim}{r}=\begin{pmatrix}1\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix}$.

(i) Write down the vector equation of the sphere. **1**

(ii) Determine whether the line is a tangent to the sphere, clearly justifying your conclusion. **3**

Please turn over for Question 14.

Question 14 (15 marks) Use a separate writing booklet.

(a) Find $\int x^2 \ln x dx$. **3**

(b) Prove by induction that $4^{n+1} + 6^n$ is divisible by 10 for all positive even integers n. **4**

(c) Find the modulus and argument of $z = \sin\frac{5\pi}{14} - i\cos\frac{9\pi}{14}$. **2**

(d) Find the locus of the complex number z satisfying $|z-1| = 2|z|$ and sketch it on the complex plane. **3**

(e) Prove by contradiction that $\log_4 7$ is irrational. **3**

Question 15 (15 marks) Use a separate writing booklet.

(a) Find $\int \frac{dx}{x^2\sqrt{x^2-9}}$. **3**

(b) (i) Prove $\frac{m+n}{2} \geq \sqrt{mn}$ for positive integers m and n. **1**

(ii) Hence prove $\frac{k+l+m+n}{4} \geq \sqrt[4]{klmn}$ for positive integers k, l, m and n. **2**

(iii) Hence prove $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ for positive integers x, y and z. **2**

(c) Solve $2z^2 + (1-i)z = i - 1$. **3**

Write your solutions in the form $z = a + bi$ where $a, b \in \mathbb{R}$

(d) With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations: **4**

$$l_1 : \underset{\sim}{r} = \left(10\underset{\sim}{i} - 9\underset{\sim}{k}\right) + \lambda\left(-\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$l_2 : \underset{\sim}{r} = \left(17\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right) + \mu\left(5\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}\right),$$

where λ and μ are scalar parameters.

Show that l_1 and l_2 meet and find the position vector of the point of intersection.

Question 16 (15 marks) Use a separate writing booklet.

(a) Let $I_n = \int_1^e (\ln x)^n \, dx$, where $n = 0,1,2,...$

(i) Show that $I_n = e - nI_{n-1}$ where $n = 1,2,...$. **2**

(ii) Hence, or otherwise, evaluate I_3. **1**

(b) (i) Show that $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$. **1**

(ii) If ω is a complex root of $z^5 - 1$, show that $1 + \omega^2 + \omega^4 = -(\omega + \omega^3)$. **1**

(iii) Hence, show that $\cos\frac{2\pi}{5} + \frac{1}{2} = \cos\frac{\pi}{5}$. **2**

(c) A particle of mass M kilograms is projected vertically upward with a velocity of $120ms^{-1}$.The air resistance acting on the particle is $3Mv$ newtons where v is the velocity of the particle.

(i) Show that if the acceleration due to gravity is $10ms^{-1}$, then the equation of motion is given by $\ddot{x} = -10 - 3v$. **1**

(ii) Find the maximum height achieved by the particle correct to the nearest centimetre. **3**

(iii) On achieving the maximum height the particle begins to drop. Find the terminal velocity as it drops. **2**

(d) Prove $|x - y| + |z - y| \geq x - z$ for all $x, y, z \in \mathbb{R}$ where $x > y > z$ **2**

End of Examination

HSC MATHEMATICS EXTENSION 2 2023
ASCHAM TRIAL SOLUTIONS

Section 1 Mutiple Choice

1. $i^{2023} + i^{2022} + i^{2021} = i^{4(505)+3} + i^{4(505)+2} + i^{4(505)+1}$

$= 1.i^3 + 1.i^2 + 1.i$
$= -i - 1 + i$
$= -1$

B

2. $P \Rightarrow Q$ has the contra positive $\neg Q \Rightarrow \neg P$

$\therefore$ contra positive is $\neg \text{girl} \Rightarrow \neg \text{chess}$

C

3. direction of $\overrightarrow{AB}$ is $\begin{pmatrix} -2-1 \\ 4-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\therefore$ equation can only be D. or $-\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

D

4. $|\underset{\sim}{u}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = |\underset{\sim}{v}|^2 = 4^2 + 2^2 + 4^2 = 36$

$\therefore |\underset{\sim}{u}| \times \underset{\sim}{v} \cdot \underset{\sim}{v} = 36\sqrt{14}$

C

5. $v = e^x \cos x = v(x)$

$a = v.\frac{dv}{dx}$

now $\frac{dv}{dx} = e^x \cos x - e^x \sin x$

$= e^x(\cos x - \sin x)$

$\therefore a = v.\frac{dv}{dx} = e^x \cos x.\ e^x(\cos x - \sin x)$

$= e^{2x} \cos x(\cos x - \sin x)$

(B)

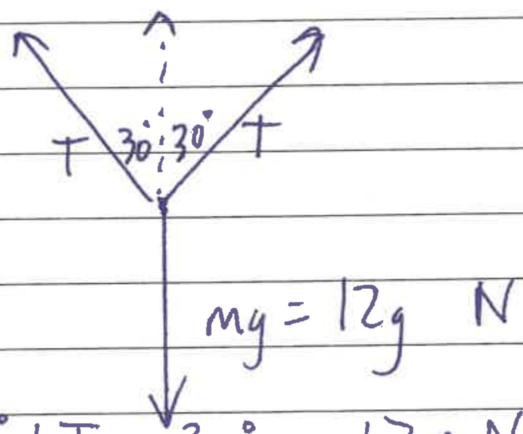
6. the statement still holds when $n=1$ or $n=0$,

$n=\frac{1}{2}$ is <u>not</u> a positive integer

$\therefore$ Statement holds.

(A)

7.



$T\cos 30° + T\cos 30° = 12g$ N

$T\left(\frac{\sqrt{3}}{2}\right) + T\left(\frac{\sqrt{3}}{2}\right) = 12g$ N

$\sqrt{3}T = 12g$ N and $T = \frac{12}{\sqrt{3}}g$ N

$= 4\sqrt{3}g$

(D)

8. $\int \frac{1}{2\tan\alpha\sqrt{4+(2\tan\alpha)^2}}\,dx = \int \frac{1}{2\tan\alpha\; 2\sec x}$

etc.

D

9. A

10. $z\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2$
$= x^2+y^2 = 2$ ✗

$z^2 = (x+iy)^2 = x^2-y^2+2xyi = 4$ ✗

$Re(z^2) + Im(z^2) = x^2-y^2+2xy = 4$ ✗

$(z+\bar{z})^2 - (z-\bar{z})^2 = (x+iy+x-iy)^2 - (x+iy-(x-iy))^2$
$= (2x)^2 - (2iy)^2$
$= 4x^2+4y^2 = 16$
$x^2+y^2 = 4$ ✓

D

Section II

11 a) $\cos\theta = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \dfrac{2\times3+-1\times2+3\times5}{\sqrt{2^2+(-1)^2+3^2}\times\sqrt{3^2+2^2+5^2}}$

$= \dfrac{19}{\sqrt{14}\times\sqrt{38}}$ ✓ $=0.82375\ldots$

$\theta = 34.537$

$= 35°$ (n. degree) ✓

b) $z^2 - z + 7 = 0$

$z = \dfrac{1 \pm \sqrt{1^2 - 4\times1\times7}}{2\times1}$ ✓

$= \dfrac{1 \pm \sqrt{27i^2}}{2}$

$z = \dfrac{1+3i\sqrt{3}}{2}$ ✓ or $\dfrac{1-3i\sqrt{3}}{2}$ ✓

c)i) $\overrightarrow{AB} = \begin{pmatrix} 4-2 \\ -2-3 \\ 3--4 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$ ✓ let $D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} d_1-4 \\ d_2--2 \\ d_3-3 \end{pmatrix} = \begin{pmatrix} d_1-4 \\ d_2+2 \\ d_3-3 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{BD} \Rightarrow \begin{pmatrix} d_1-4=2 \\ d_2+2=-5 \\ d_3-3=7 \end{pmatrix}$

$\therefore d_1 = 6,\ d_2 = -7,\ d_3 = 10$

and position vector D is: $6\underset{\sim}{i} - 7\underset{\sim}{j} + 10\underset{\sim}{k}$. ✓

11 c(ii) $|\vec{AC}| = 4$

$$\vec{AC} = \begin{pmatrix} a-2 \\ 5-3 \\ -2--4 \end{pmatrix} = \begin{pmatrix} a-2 \\ 2 \\ 2 \end{pmatrix} \checkmark$$

$$|\vec{AC}|^2 = (a-2)^2 + 2^2 + 2^2 = 4^2$$

$$\therefore a^2 - 4a + 4 + 4 + 4 = 4 \times 4$$

$$a^2 - 4a = 4 \checkmark$$

$$a^2 - 4a - 4 = 0$$

$$a^2 - 4a + 4 = 8$$

$$(a-2)^2 = 4 \times 2$$

$$a - 2 = \pm 2\sqrt{2}$$

$$a = 2 \pm 2\sqrt{2}$$

but $a > 0$ and so $a = 2 + 2\sqrt{2}$ ✓

(d) i) $v^2 = 4(2x - x^2)$

$$\tfrac{1}{2}v^2 = 2(2x - x^2) = 4x - 2x^2 \checkmark$$

$$a = \frac{d\left(\tfrac{1}{2}v^2\right)}{dx} = 4 - 4x \checkmark$$

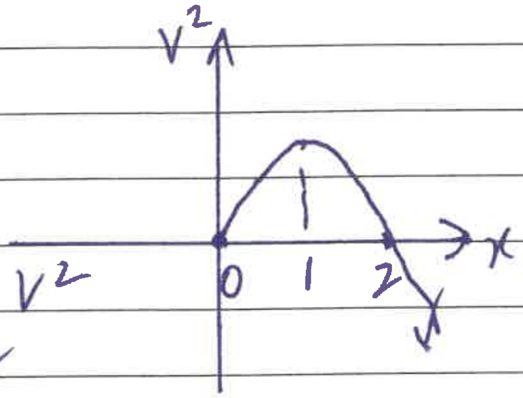
$$= 4(1-x) = -4x + 4$$

$$= -4(x-1) \text{ as required}$$

ii) The particle moves in SHM since its acceleration formula can be written in the form: $\ddot{x} = -n^2(x-c)$ ✓ where $\ddot{x} = a$, $n = 2$, $c = 1$.

(iii) max speed occurs for max v^2.

$$v^2 = 8x - 4x^2$$
$$= 4x(2-x)$$



by symmetry, max v^2 is when $x=1$. ✓

$$\therefore \max v^2 = 4\times 1(2-1)$$
$$= 4$$

and max speed $= 2$ ✓

12 a) i) let $\dfrac{x^2+1}{(x-1)(x+2)(x^2+x+1)} = \dfrac{a}{x-1} + \dfrac{b}{x+2} + \dfrac{cx+d}{x^2+x+1}$

$$\therefore x^2+1 = a(x+2)(x^2+x+1) + b(x-1)(x^2+x+1) + (cx+d)(x-1)(x+2) \quad (1)$$

sub $x=-2$ into (1):

$$(-2)^2+1 = b(-2-1)((-2)^2+-2+1)$$
$$5 = b(-3)(3) \qquad \therefore b = -\frac{5}{9} \checkmark$$

sub $x=1$ into (1):

$$1^2+1 = a(1+2)(1^2+1+1)$$
$$2 = a(3)(3) \qquad \therefore a = \frac{2}{9} \checkmark$$

Examine coefficients of x^3 on both sides of (1):

$$0x^3 = ax^3 + bx^3 + cx^3$$

12(a)i) cont.

$\therefore\ a+b+c=0$

or $\frac{2}{9}+\frac{-5}{9}+c=0$

$\therefore c=\frac{3}{9}=\frac{1}{3}$ ✓

examine constant terms by setting $x=0$ in (1):

$1=a(2)(1)+b(-1)(1)+d(-1)(2)$

$1=2a-b-2d$

or $1=2\left(\frac{2}{9}\right)-\left(-\frac{5}{9}\right)-2d$

$2d=\frac{4}{9}+\frac{5}{9}-1=0$

$\therefore d=0$ ✓

$$\frac{x^2+1}{(x-1)(x+2)(x^2+x+1)}=\frac{2}{9(x-1)}-\frac{5}{9(x+2)}+\frac{x}{3(x^2+x+1)}$$

ii) $$\int\frac{x^2+1}{(x-1)(x+2)(x^2+x+1)}dx=\frac{2}{9}\int\frac{dx}{(x-1)}-\frac{5}{9}\int\frac{dx}{x+2}$$

$$+\frac{1}{3}\int\frac{x}{x^2+x+1}dx$$

Now $\int\frac{x}{x^2+x+1}dx=\frac{1}{2}\int\frac{2x}{x^2+x+1}dx=\frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx$

$$-\frac{1}{2}\int\frac{1}{x^2+x+1}dx$$

so $\displaystyle\int \frac{x}{x^2+x+1}\,dx = \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{(\frac{\sqrt{3}}{2})}\right)\right)$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) ✓ ✓$$

Hence $\displaystyle\int \frac{x^2+1}{(x-1)(x+2)(x^2+x+1)}\,dx$

$$= \frac{2}{9}\ln|x-1| - \frac{5}{9}\ln|x+2| + \frac{1}{6}\ln|x^2+x+1| ✓$$

$$-\frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

12(b) $\quad 1 < x < 11$ ✓

12(c) $\displaystyle \frac{3+i}{7-2i} \times \frac{7+2i}{7+2i} = \frac{21-2+6i+7i}{49+4}$ ✓

$$= \frac{19}{53} + \frac{13}{53}i$$

Real part is $\frac{19}{53}$ ✓ and imaginary part is $\frac{13}{53}$ ✓

12 (d)

$$2z^6 + z^4 - 2z^2 - 1 = 0$$

let $z^2 = u$

let $P(u) = 2u^3 + u^2 - 2u - 1 = 0$

$P(1) = 2 + 1 - 2 - 1 = 0 \quad \therefore u - 1$ is a factor

$P(-1) = -2 + 1 + 2 - 1 = 0 \quad \therefore u + 1$ is a factor.

$(u-1)(u+1)(2u+1) = 2u^3 + u^2 - 2u - 1$

by observation

Check: $(2u+1)(u^2-1) = 2u^3 + u^2 - 2u - 1$ as required.

$\therefore 2z^6 + z^4 - 2z^2 - 1 = (2z^2+1)(z^2-1)(z^2+1)$ ✓

$= (2z^2 - i^2)(z-1)(z+1)(z^2 - i^2)$

$= (\sqrt{2}z - i)(\sqrt{2}z + i)(z-1)(z+1)(z-i)(z+i)$

✓ ✓ ✓

13(a) $\int \sec x \, dx = \int \frac{1}{\cos x} dx$

now $\cos x = \frac{1-t^2}{1+t^2}$ given $\tan\frac{x}{2} = t$

and $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2\cos^2\frac{x}{2}} = \frac{1}{2\left(\frac{1}{t^2+1}\right)}$

$\frac{dt}{dx} = \frac{t^2+1}{2}$ and $\frac{dx}{dt} = \frac{2}{t^2+1}$ or $dx = \frac{2dt}{1+t^2}$

$\therefore \int \sec x \, dx = \int \frac{1+t^2}{1-t^2}\left(\frac{2}{1+t^2}\right) dt$

$$\therefore \int \sec x\,dx = \int \frac{2}{1-t^2}\,dt \checkmark$$

now $\dfrac{2}{(1-t)(1+t)} = \dfrac{a}{1-t} + \dfrac{b}{1+t}$

$a(1+t) + b(1-t) = 2 \Rightarrow a+b = 2$

$a + b + t(a-b) = 2$ and $a-b=0$

$\therefore a = b = 1$

$$\therefore \int \sec x\,dx = \int \frac{1}{1+t}\,dt + \int \frac{-1}{1-t}\,dt \checkmark$$

$$= \ln|1+t| - \ln|1-t|$$

$$= \ln\left|1+\tan\left(\tfrac{x}{2}\right)\right| - \ln\left|1-\tan\left(\tfrac{x}{2}\right)\right| + c$$

$\checkmark$

13(bi) let $z = \sqrt{3} + i = r\,\text{cis}\,\theta$

$= r\cos\theta + ir\sin\theta$

$\therefore r\cos\theta = \sqrt{3}$, and $r\sin\theta = 1$

$r^2\cos^2\theta + r^2\sin^2\theta = 3 + 1 = 4$

$r^2(\sin^2\theta + \cos^2\theta) = 4$

$r^2 = 4$

$r = 2 \checkmark$

$2\cos\theta = \sqrt{3}$ $\qquad$ $2\sin\theta = 1$

$\cos\theta = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$\therefore \sqrt{3} + i = 2\,\text{cis}\,\frac{\pi}{6} = 2e^{i\left(\frac{\pi}{6}\right)} \checkmark$$

13(b)(ii) $z^{18} = \left(2e^{i\frac{\pi}{6}}\right)^{18}$

$= 2^{18} e^{i\left(\frac{18\pi}{6}\right)}$

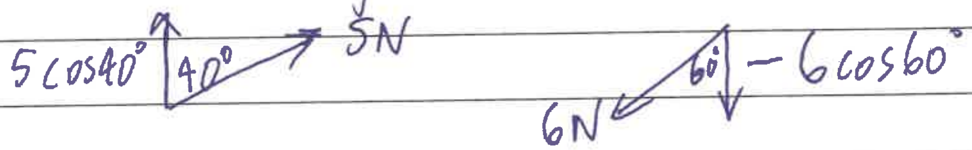
$= 2^{18} e^{i(3\pi)}$

$= 2^{18}(-1)$ ✓

$= -2^{18}$

$= -262144$ ✓

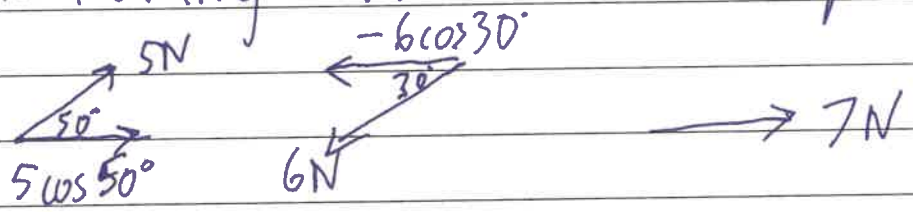
13(c) vertically: assume ↑ is positive.



$5\cos 40° - 6\cos 60° = 5\cos 40° - 3$ ✓

≈ 0.83 (3 sig figs)

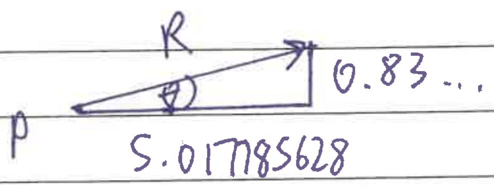
horizontally: assume → is positive



$7 + 5\cos 50° - 6\cos 30° \approx 5.017785628$

✓

Resultant



$R^2 = (0.83..)^2 + (5.01778..)^2$

$R = 5.086$N (4 sig figs) ✓

$\tan\theta = \dfrac{0.83...}{5.017785...}$

$\tan\theta = 0.165456..$ and $\theta = 9°$ (n. degree) ✓

13 (d)(i) $(x-3)^2 + (y+3)^2 + (z-4)^2 = 6^2 = 36$ ✓ 1

(ii) $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 5-\lambda \\ 4-\lambda \end{pmatrix}$

sub $x = 1+2\lambda$, $y = 5-\lambda$ and $z = 4-\lambda$ into 1:

$(1+2\lambda-3)^2 + (5-\lambda+3)^2 + (4-\lambda-4)^2 = 36$

$(2(\lambda-1))^2 + (8-\lambda)^2 + (-\lambda)^2 = 36$

$4(\lambda^2-2\lambda+1) + 64 - 16\lambda + \lambda^2 + \lambda^2 = 36$

$4\lambda^2 + 2\lambda^2 - 8\lambda - 16\lambda + 4 + 64 = 36$

$6\lambda^2 - 24\lambda + 68 - 36 = 0$

$6\lambda^2 - 24\lambda + 32 = 0$ ✓

$3\lambda^2 - 12\lambda + 16 = 0$

for λ, check $\Delta = (-12)^2 - 4\times3\times16$

$= 144 - 192$

$= -48 < 0$

$\therefore$ there is no real solution for λ ✓

and so the line and sphere do not touch $\therefore$ the line is not a tangent to the sphere. ✓

14a $\int x^2 \ln x \, dx$

let $u = \ln x$ and $dv = x^2 dx$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{x^3}{3}$ ✓

$$\int x^2 \ln x \, dx = \int u \, dv$$

$$= [uv] - \int v \, du$$

$$= \left[\frac{x^3 \ln x}{3}\right] - \int \frac{x^3}{3}\left(\frac{1}{x}\right) dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2 \, dx$$

$$= \frac{x^3 \ln x}{3} - \frac{1}{3} \cdot \frac{x^3}{3} + C$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$$ ✓ ✓

(b) let proposition be: $P(n)$: $4^{n+1} + 6^n$ is divisible by 10 for $n = 2, 4, 6, \ldots$

i.e. $P(n)$: $4^{n+1} + 6^n = 10m$; $m \in \mathbb{Z}$ and $n = 2, 4, 6, \ldots$

Step 1 Prove $P(2)$ holds.

$LHS = 4^{2+1} + 6^2 = 64 + 36 = 100 = 10m$, $(m = 10)$

$= RHS$ ✓

Step 2 Assume $P(k)$ holds

i.e. $4^{k+1} + 6^{k} = 10m$ where $m \in Z$
and $k = 2, 4, 6, \ldots$

RTP: $P(k+2)$ holds

i.e. $4^{k+2+1} + 6^{k+2} = 10p$ where $p \in Z$
and $k = 2, 4, 6, \ldots$

$LHS = 4^{k+2+1} + 6^{k+2}$

$= 4^2 \times 4^{k+1} + 6^2 \times 6^{k}$
$= 16 \times 4^{k+1} + 36 \times 6^{k}$

$= 16 \times 4^{k+1} + 16 \times 6^{k} + 20 \times 6^{k}$
$= 16(4^{k+1} + 6^{k}) + 20 \times 6^{k}$
$= 16(10m) + 20 \times 6^{k}$ (from $P(k)$ assumption)

$= 10(16m + 2 \times 6^{k})$
$= 10p$ $\because 16m + 2 \times 6^{k} \in Z$ as required.

$\therefore P(k+2)$ holds based on assuming $P(k)$ holds.

Step 3 Since $P(2)$ holds, and $P(k+2)$ holds given $P(k)$ holds, by the process of mathematical induction,

$P(n)$ holds for all $n = 2, 4, 6, \ldots$

14 (c) $z = \sin\frac{5\pi}{14} - i\cos\frac{9\pi}{14}$

$= \cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right) - i\sin\left(\frac{\pi}{2} - \frac{9\pi}{14}\right)$

$= \cos\left(\frac{7\pi}{14} - \frac{5\pi}{14}\right) - i\sin\left(\frac{7\pi}{14} - \frac{9\pi}{14}\right)$

$= \cos\left(\frac{2\pi}{14}\right) - i\sin\left(-\frac{2\pi}{14}\right)$

$= \cos\left(\frac{\pi}{7}\right) - i\sin\left(-\frac{\pi}{7}\right) = \cos\frac{\pi}{7} - i\left(-\sin\frac{\pi}{7}\right)$

$= \cos\left(\frac{\pi}{7}\right) + i\sin\left(\frac{\pi}{7}\right)$

$= 1\,\text{cis}\left(\frac{\pi}{7}\right)$

$\therefore$ modulus $= 1$ ✓ and argument $= \frac{\pi}{7}$ ✓

(d) $|z-1| = 2|z|$

let $z = x+iy$ $\quad |z-1| = |(x-1)+iy| = \sqrt{(x-1)^2+y^2}$

$2|z| = 2\sqrt{x^2+y^2}$

$\therefore \left(\sqrt{(x-1)^2+y^2}\right)^2 = \left(2\sqrt{x^2+y^2}\right)^2$ ✓

$(x-1)^2 + y^2 = 4(x^2+y^2)$

$x^2-2x+1+y^2 = 4x^2+4y^2$

$0 = 3x^2+2x-1+3y^2$

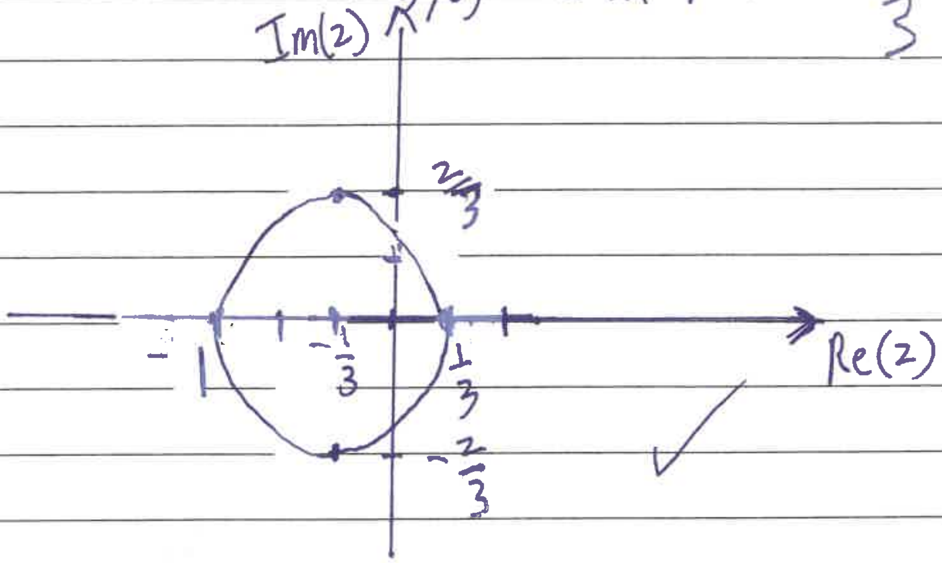
$1 = 3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)+3y^2$

$1 = -3\times\frac{1}{9} + 3\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)+3y^2$

$1+\frac{1}{3} = 3\left(x+\frac{1}{3}\right)^2+3y^2$

$\left(x+\frac{1}{3}\right)^2 + y^2 = \left(\frac{2}{3}\right)^2$ ✓

circle with centre $(-\frac{1}{3}, 0)$ and radius $\frac{2}{3}$



(e) Assume $\log_4 7$ is rational (1)

i.e. $\log_4 7 = \frac{a}{b}$ where $a, b \in \mathbb{Z}$.

$$4^{\log_4 7} = 4^{\frac{a}{b}}$$

$$7 = 4^{\frac{a}{b}}$$

$$7^b = 4^a$$

Powers of 7 end in 7, 9, 3, 1

but powers of 4 are all even.

$\therefore$ there are no integer powers of 7 that equal integer powers of 4.

Hence our assumption (1) does not hold true due to contradiction and $\log_4 7$ is irrational.

15a) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}}$

let $x = 3\sec\theta = 3(\cos\theta)^{-1}$ ✓

$\dfrac{dx}{d\theta} = -\sin\theta \times -3(\cos\theta)^{-2}$

$= 3\,\dfrac{\sin\theta}{\cos\theta} \times \dfrac{1}{\cos\theta} = 3\sec\theta\tan\theta$

$\therefore dx = 3\sec\theta\tan\theta\,d\theta$

also $\sqrt{x^2-9} = \sqrt{(3\sec\theta)^2-9} = \sqrt{9(\sec^2\theta-1)}$

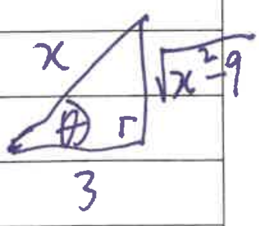
$= 3\sqrt{\tan^2\theta} = 3\tan\theta$

$\therefore \displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{3^2\sec^2\theta \times 3\tan\theta}$

$= \dfrac{1}{9}\displaystyle\int \cos\theta\,d\theta$

$= \dfrac{1}{9}\sin\theta$ ✓

now $\sec\theta = \dfrac{x}{3}$ $\quad\therefore \cos\theta = \dfrac{3}{x}$



$\therefore \displaystyle\int \frac{dx}{x^2\sqrt{x^2-9}} = \frac{1}{9}\sin\theta + C$

$= \dfrac{\sqrt{x^2-9}}{9x} + C$ ✓

$\left(\because \sin\theta = \dfrac{\sqrt{x^2-9}}{x}\right)$

15 b) i) $(\sqrt{m}-\sqrt{n})^2 > 0$

$$m + n - 2\sqrt{m}\sqrt{n} > 0$$

$$m+n > 2\sqrt{m}\sqrt{n}$$ ✓

$$\frac{m+n}{2} > \sqrt{mn}$$

ii) from i) $\frac{m+n}{2} > \sqrt{mn}$

$\therefore \frac{k+l}{2} > \sqrt{kl}$

and $$\frac{\frac{k+l}{2} + \frac{m+n}{2}}{2} > \sqrt{\left(\frac{k+l}{2}\right)\left(\frac{m+n}{2}\right)}$$

or $$\frac{k+l+m+n}{4} > \sqrt{\left(\frac{k+l}{2}\right)\left(\frac{m+n}{2}\right)}$$ ✓

but $$\sqrt{\left(\frac{k+l}{2}\right)\left(\frac{m+n}{2}\right)} > \sqrt{\sqrt{kl}\sqrt{mn}}$$ ✓

$$> \sqrt[4]{klmn}$$

$\therefore \frac{k+l+m+n}{4} > \sqrt[4]{klmn}$ as required

15(b) iii) substitute $k=x$, $l=y$, $m=z$

and $n = \frac{x+y+z}{3}$

into part ii) above.

i.e. $$\frac{x+y+z+\frac{x+y+z}{3}}{4} \geq \left(xyz\left(\frac{x+y+z}{3}\right)\right)^{1/4}$$

$$\frac{\left(\frac{3x+3y+3z+x+y+z}{3}\right)}{4} \geq \left(xyz\left(\frac{x+y+z}{3}\right)\right)^{1/4}$$

$$\frac{4x+4y+4z}{12} \geq \left(xyz\left(\frac{x+y+z}{3}\right)\right)^{1/4}$$

$$\frac{x+y+z}{3} \geq (xyz)^{1/4}\left(\frac{x+y+z}{3}\right)^{1/4}$$

$$\left(\frac{x+y+z}{3}\right)^4 \geq xyz\left(\frac{x+y+z}{3}\right)$$

$$\left(\frac{x+y+z}{3}\right)^{4-1} \geq xyz$$

$$\left(\frac{x+y+z}{3}\right)^3 \geq xyz. \checkmark$$

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$$ as required.

15 c) $2z^2 + (1-i)z = i - 1$

$2z^2 + (1-i)z - (i-1) = 0$

$2z^2 + (1-i)z + (1-i) = 0$

$z = \dfrac{-(1-i) \pm \sqrt{(1-i)^2 - 4\times2(1-i)}}{2\times2} = \dfrac{(i-1) \pm \sqrt{(1-i)^2 - 8(1-i)}}{4}$ ✓

now $\sqrt{(1-i)^2 - 8(1-i)} = \sqrt{1 - 2i + i^2 - 8 + 8i}$

$= \sqrt{6i - 8}$

let $(x+iy)^2 = 6i - 8$

$x^2 - y^2 = -8$ and $2xyi = 6i$

by inspection $x = \pm 1$, $y = \pm 3$

$\therefore \sqrt{6i-8} = -1-3i$ or $1+3i$

$\therefore z = \dfrac{i-1+(-1-3i)}{4}$ or $\dfrac{i-1-(-1-3i)}{4}$ or $\dfrac{i-1+1+3i}{4}$ or $\dfrac{i-1-(1+3i)}{4}$

$= \dfrac{-2-2i}{4}, \dfrac{4i}{4}, \dfrac{4i}{4}, \dfrac{-2-2i}{4}$

$z = i$ or $\dfrac{-1-i}{2}$ ✓ ✓

15 (d) $l_1 : \underset{\sim}{r} = \begin{pmatrix} 10-\lambda \\ \lambda \\ -9+2\lambda \end{pmatrix}$ $\quad l_2 : \underset{\sim}{r} = \begin{pmatrix} 17+5\mu \\ 1-\mu \\ 3+3\mu \end{pmatrix}$

for l_1 and l_2 to cross, solve for λ and μ.

$10-\lambda = 17+5\mu$ ① and $\lambda = 1-\mu$ ②

Sub ② into ①:

$10-(1-\mu) = 17+5\mu$

$9-17 = 4\mu$

$-8 = 4\mu$ so $\mu = -2$ ✓

and $\lambda = 1-(-2)$ (from ②)

$=3$ ✓

Check: $-9+2\lambda = 3+3\mu$ ✓

when $\lambda = 3$, LHS $= -9+2\times3 = -3$

$\mu = -2$, RHS $= 3+3\times-2 = -3 =$ LHS

$\therefore$ l_1 and l_2 cross at: $\begin{pmatrix} 10-3 \\ 3 \\ -9+2\times3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix}$ ✓

$7\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}$

16(a) $I_n = \int_1^e (\ln x)^n dx$

i) Let $\int (\ln x)^n dx = \int u\, dv$

where $u = (\ln x)^n$ and $1 = dv$

$\frac{du}{dx} = n(\ln x)^{n-1} \times \frac{1}{x}$ $\quad x = v$

$\therefore du = \frac{n}{x}(\ln x)^{n-1} dx$ ✓

$\therefore \int_1^e (\ln x)^n dx = \left[(\ln x)^n \times x\right]_1^e - \int_1^e x\left(\frac{n}{x}(\ln x)^{n-1} dx\right)$

$= (\ln e)^n \times e - n\int_1^e (\ln x)^{n-1} dx$ ✓

$= 1^n \times e - n\, I_{n-1}$

$\therefore I_n = e - n\, I_{n-1}$ as required.

ii) $I_0 = \int_1^e (\ln x)^0 dx = \int_1^e 1\, dx$

$= [x]_1^e$

$= e - 1$

16 (a)(ii) cont.

$$I_3 = e - 3I_2 = e - 3(e - 2I_1)$$
$$= e - 3(e - 2(e - I_0))$$
$$= e - 3(e - 2(e - (e-1)))$$
$$= e - 3(e - 2)$$
$$= e - 3e + 6$$
$$= 6 - 2e \checkmark$$

(b) $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

(i) $RHS = (z-1)(z^4 + z^3 + z^2 + z + 1)$

$$= z^5 + z^4 + z^3 + z^2 + z$$
$$\quad - z^4 - z^3 - z^2 - z - 1$$
$$= z^5 - 1$$ as required.

ii) If ω is a root of $z^5 - 1$,
then $\therefore P(z) = z^5 - 1 \Rightarrow P(\omega) = 0$

ie. $\omega^5 - 1 = 0$

or $(\omega - 1)(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0 \checkmark$

$\therefore \omega = 1$ or $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$

$1 + \omega^2 + \omega^4 = -(\omega + \omega^3)$

as required.

iii) $z^5 = 1$

let $z = \text{cis}\,\theta$

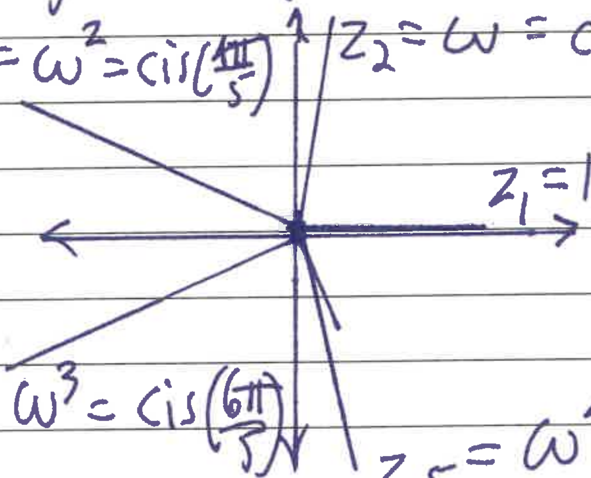
$z^5 = \text{cis}(5\theta) = 1 + 0i$

$\therefore \cos 5\theta = 1$ and $\sin 5\theta = 0$

$5\theta = 0$ or 2π

$\theta = \frac{2\pi}{5}$ or 0.

$\therefore$ the roots are spaced $\frac{2\pi}{5}$ around the Argand diagram starting at $z=1$



$z_3 = \omega^2 = \text{cis}\left(\frac{4\pi}{5}\right)$

$z_2 = \omega = \text{cis}\left(\frac{2\pi}{5}\right)$

$z_1 = 1$

$z_4 = \omega^3 = \text{cis}\left(\frac{6\pi}{5}\right) = \text{cis}\left(-\frac{4\pi}{5}\right)$

$z_5 = \omega^4 = \text{cis}\left(\frac{8\pi}{5}\right) = \text{cis}\left(-\frac{2\pi}{5}\right)$ ✓

we know $1 + \omega^2 + \omega^4 = -\omega - \omega^3$ (from ii)

$1 + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) + \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$

$= -\cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} - \cos\left(-\frac{4\pi}{5}\right) - i\sin\left(-\frac{4\pi}{5}\right)$

equating real parts:

$1 + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) = -\cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{4\pi}{5}\right)$

$1 + 2\cos\frac{4\pi}{5} = -2\cos\frac{2\pi}{5} \Rightarrow \frac{1}{2} + \cos\frac{2\pi}{5} = -\cos\frac{4\pi}{5}$

now $-\cos\frac{4\pi}{5} = -\left(-\cos\left(\pi - \frac{4\pi}{5}\right)\right) = \cos\frac{\pi}{5}$ ✓

$\therefore \cos\frac{2\pi}{5} + \frac{1}{2} = \cos\frac{\pi}{5}$ as required.

16 (c) i)

$$M\ddot{x} = -10M - 3Mv$$
$$\ddot{x} = -10 - 3v$$

ii) max height = max x value.

$$\ddot{x} = -10 - 3v$$
$$v\frac{dv}{dx} = -10 - 3v$$
$$\frac{dv}{dx} = \frac{-10-3v}{v}$$
$$\frac{dx}{dv} = \frac{v}{-10-3v}$$
$$x = \int \frac{v}{-10-3v}\,dv = \frac{1}{3}\int \frac{-3v}{-10-3v}\,dv$$
$$= -\frac{1}{3}\int \frac{3v}{3v+10}\,dv = -\frac{1}{3}\int\left(\frac{3v+10}{3v+10} - \frac{10}{3v+10}\right)dv$$
$$= -\frac{1}{3}\int\left(1 - \frac{10}{3v+10}\right)dv$$
$$= -\frac{1}{3}\int 1\,dv + \frac{1}{3}\int \frac{10}{3v+10}\,dv$$
$$= -\frac{1}{3}v + \frac{10}{9}\int \frac{3}{3v+10}\,dv$$
$$= -\frac{1}{3}v + \frac{10}{9}\ln|3v+10| + C$$

when $x=0$, $v=120$

$\therefore 0 = -\frac{1}{3}\times 120 + \frac{10}{9}\ln|3\times120+10| + c$

$= -40 + \frac{10}{9}\ln|370| + c$

$\therefore c = 40 - \frac{10}{9}\ln(370)$

$x = -\frac{1}{3}v + \frac{10}{9}\ln|3v+10| + 40 - \frac{10}{9}\ln(370)$

max height is when $v=0$.

i.e. $x = \frac{10}{9}\ln|10| - \frac{10}{9}\ln(370) + 40$

$= 40 + \frac{10}{9}\ln\left(\frac{10}{370}\right) = 40 - \frac{10}{9}\ln 37$

$= 35.987869\ldots$ m

$= 3599$ cm (n. cm) ✓

16c)(iii) terminal velocity occurs with $a=0$.

Now with downward motion, resistance is in opposite direction to the motion.

assume $\downarrow$ is positive // $\therefore \ddot{x} = 10 - 3v$ ✓

so when $\ddot{x}=0$, $3v=10$

$v = \frac{10}{3}$ m/s is the terminal velocity. ✓

16(d) RTP: $|x-y|+|z-y| \geqslant x-z \quad \forall x,y,z \in \mathbb{R}$
and $x > y > z$

By the triangle inequality

$$|a|+|b| \geqslant |a+b|$$

Also, we know that $|a| \geqslant a \Rightarrow |a+b| \geqslant a+b$

Further $|z-y| = |-y+z|$
$= |-(y-z)|$
$= |y-z|$

So $|x-y|+|z-y| = |x-y|+|y-z|$

$|x-y|+|y-z| \geqslant |x-y+y-z| = |x-z|$

but $|x-z| \geqslant x-z$ (recall $x > z$)

$\therefore |x-y|+|z-y| \geqslant |x-z| \geqslant x-z$
as required.